Integral and Nonnegativity Preserving Approximations of Functions

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Abstract

In this paper we consider the problem of approximating a function by continuous piecewise linear functions that preserve the integral and nonnegativity of the original function.

Key words: Integral preserving, Positivity preserving

1 Introduction

The problem of approximating a function by piecewise polynomials is central in many branches of mathematics. In this paper we consider the following problem: given a finite uniform partition of the unit interval $I = [0, 1]$ or the unit square $I \times I = [0, 1] \times [0, 1]$, find a continuous piecewise linear function that is integral and nonnegativity preserving for every integrable function. This problem has applications in, e.g., the numerical analysis of Markov operators in stochastic analysis and Frobenius-Perron operators in ergodic theory [2]. For example, the famous Ulam conjecture [6], [5] is related to integral and nonnegativity preserving approximations via piecewise constant functions.

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Preprint submitted to Elsevier Science 25 March 2006
In the next section we give two results for $L^1$ spaces. Then in Sections 3 and 4 we concentrate on the context of $L^1(I)$ and $L^1(I \times I)$, respectively.

2 Some Averaging Operators on $L^1(X)$

Let $(X, \mathcal{A}, P)$ be a probability space and let $\psi_0, \psi_1, \ldots, \psi_m$ be nonnegative $\mathcal{A}$-measurable functions on $X$ such that $\psi_0 + \psi_1 + \cdots + \psi_m = 1$. Assume that $\psi_0, \psi_1, \ldots, \psi_m$ are linearly independent in $L^1(X)$ and let $\Psi_m$ denote the linear span of $\psi_0, \psi_1, \ldots, \psi_m$ in $L^1(X)$.

Let $T$ be a continuous linear operator from $L^1(X)$ to $\Psi_m$. Given $f \in L^1(X)$ and $g \in L^\infty(X)$, define $\langle f, g \rangle = \int_X fg \, dP$. Since the dual of $L^1(X)$ is $L^\infty(X)$, there exist $w_0, w_1, \ldots, w_m \in L^\infty(X)$ such that

$$T(f) = \sum_{i=0}^m \langle f, w_i \rangle \psi_i \quad \text{for each} \quad f \in L^1(X).$$

$T$ is called nonnegative if $T$ maps nonnegative functions to nonnegative functions. We say that $T$ preserves integrals if $\int_X T(f) \, dP = \int_X f \, dP$ for each $f \in L^1(X)$. We say that $T$ is an averaging operator from $L^1(X)$ to $\Psi_m$ if $T(1) = 1$ and if $T$ is nonnegative and preserves integrals.

Theorem 1 Let $\psi_0, \psi_1, \ldots, \psi_m$ and $\Psi_m$ be as above. Let $w_0, w_1, \ldots, w_m \in L^\infty(X)$ and define $T : L^1(X) \to \Psi_m$ by

$$T(f) = \sum_{i=0}^m \langle f, w_i \rangle \psi_i \quad \text{for each} \quad f \in L^1(X).$$

(1) $T(1) = 1$ if and only if $\langle 1, w_i \rangle = 1$ for $i = 0, 1, \ldots, m$.

(2) $T$ is nonnegative if $w_i \geq 0$ a.e. for $i = 0, 1, \ldots, m$.

(3) $T$ preserves integrals if and only if $\sum_{i=0}^m \langle \psi_i, 1 \rangle w_i = 1$ a.e.

Proof. (1): Suppose $T(1) = 1$. Then $\sum_{i=0}^m \psi_i = 1 = T(1) = \sum_{i=0}^m \langle 1, w_i \rangle \psi_i$. Hence, $\langle 1, w_i \rangle = 1$ for $i = 0, 1, \ldots, m$. The converse is clear. (2): Clearly $T$ is nonnegative if $w_i \geq 0$ a.e. for $i = 0, 1, \ldots, m$. (3): If $f \in L^1(X)$, then

$$\langle T(f), 1 \rangle = \left\langle \sum_{i=0}^m \langle f, w_i \rangle \psi_i, 1 \right\rangle = \sum_{i=0}^m \langle f, w_i \rangle \langle \psi_i, 1 \rangle = \left\langle f, \sum_{i=0}^m \langle \psi_i, 1 \rangle w_i \right\rangle.$$

Hence, $\sum_{i=0}^m \langle \psi_i, 1 \rangle w_i = 1$ a.e. if and only if $\langle T(f), 1 \rangle = \langle f, 1 \rangle$ for each $f$ in $L^1(X)$. □
Note. Let $X = I$ with the Lebesgue measure. Suppose $\psi_0(x) = 1 - x/2$ and $\psi_1(x) = x/2$. If we choose $w_0 = 1$ and $w_1 = -1$, then $Tf(x) = \langle f, 1 \rangle (1 - x)$ and so $T$ is nonnegative. Thus $T$ is nonnegative does not imply that $w_i \geq 0$ for all $i$.

**Theorem 2** Let $\psi_0, \psi_1, \ldots, \psi_m$ and $\Psi_m$ be as above. Let $V_0, V_1, \ldots, V_m \in \mathcal{A}$ such that $P(V_i) > 0$ for each $i = 0, 1, \ldots, m$. Define $Q : L^1(X) \to \Psi_m$ by

$$Q(f) = \sum_{i=0}^m \left\langle f, \frac{1}{P(V_i)} \chi_{V_i} \right\rangle \psi_i \text{ for each } f \in L^1(X).$$

Assume $P(V_i \cap V_j) = 0$ if $i \neq j$. Then $Q$ is an averaging operator from $L^1(X)$ to $\Psi_m$ if and only if $\langle \psi_k, 1 \rangle = P(V_k)$ for each $k = 0, 1, \ldots, m$.

**PROOF.** Set $w_i = \frac{1}{P(V_i)} \chi_{V_i}$ for $i = 0, 1, \ldots, m$. Assume $Q$ is an averaging operator from $L^1(X)$ to $\Psi_m$. By Theorem 1(3), $\sum_{i=0}^m \langle \psi_i, 1 \rangle w_i = 1$ a.e. Thus, for $k = 0, 1, \ldots, m$, we have

$$P(V_k) = \langle \chi_{V_k}, 1 \rangle = \sum_{i=0}^m \langle \chi_{V_i}, w_i \rangle \langle \psi_i, 1 \rangle = \langle \psi_k, 1 \rangle.$$

Assume $\langle \psi_k, 1 \rangle = P(V_k)$ for each $k = 0, 1, \ldots, m$. Then

$$1 = \langle 1, 1 \rangle = \left\langle \sum_{k=0}^m \psi_k, 1 \right\rangle = \sum_{k=0}^m \langle \psi_k, 1 \rangle = \sum_{k=0}^m P(V_k).$$

Since $P(V_i \cap V_j) = 0$ for $i \neq j$, it follows that $\sum_{k=0}^m \chi_{V_k} = 1$ a.e. and so $\sum_{i=0}^m \langle \psi_i, 1 \rangle w_i = \sum_{i=0}^m \chi_{V_i} = 1$ a.e. Thus $Q$ preserves integrals by Theorem 1(3). □

### 3 Some Averaging Operators on $L^1(I)$

Divide $I = [0, 1]$ into $n$ equal subintervals $I_i = [x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$. Let $h = 1/n = m (I_i)$, where $m$ is the Lebesgue measure. Let $\Phi_n$ denote the space of all continuous piecewise linear functions associated with the partition $0 = x_0 < x_1 < \cdots < x_n = 1$. Let $\varphi_i$ be the unique function in $\Phi_n$ such that $\varphi_i$ is 1 at the node $x_i$ and 0 at all other node points. The $(n + 1)$ nodal functions $\{\varphi_i\}_{i=0}^n$ form a canonical basis for $\Phi_n$.

Let $T$ be a continuous linear operator from $L^1(I)$ to $\Phi_n$. There exist $w_i \in L^\infty(I)$ for $i = 0, 1, \ldots, n$ such that

$$T(f) = \sum_{i=0}^n \langle f, w_i \rangle \varphi_i \text{ for each } f \in L^1(I).$$
Theorem 3 Let \( w_0, w_1, \ldots, w_n \in L^\infty(I) \) and define \( T : L^1(I) \to \Phi_n \) by

\[
T(f) = \sum_{i=0}^{n} \langle f, w_i \rangle \varphi_i \quad \text{for each} \quad f \in L^1(I).
\]

(1) \( T(1) = 1 \) if and only if \( \langle 1, w_i \rangle = 1 \) for \( i = 0, 1, \ldots, n \).

(2) \( T \) is nonnegative if and only if \( w_i \geq 0 \) a.e. for \( i = 0, 1, \ldots, n \).

(3) \( T \) preserves integrals if and only if \( w_0 + 2 \sum_{i=1}^{n-1} w_i + w_n = 2n \) a.e., or equivalently, \( \frac{1}{n} \sum_{i=1}^{n} (w_{i-1} + w_i)/2 = 1 \) a.e.

**PROOF.** Parts (1) and (3) follow from Theorem 1. In part (3), we need to use \( \langle \varphi_0, 1 \rangle = \langle \varphi_n, 1 \rangle = 1/2n \) and \( \langle \varphi_i, 1 \rangle = 1/n \) for \( 1 \leq i \leq n-1 \). Clearly \( T \) is nonnegative if \( w_i \geq 0 \) a.e. for \( i = 0, 1, \ldots, n \). Suppose \( T \) is nonnegative. Let \( A_i = \{ x : w_i(x) < 0 \} \). Then

\[
0 \leq T(\chi_{A_i})(x_i) = \sum_{j=0}^{n} \langle \chi_{A_i}, w_j \rangle \varphi_j(x_i) = \langle \chi_{A_i}, w_i \rangle.
\]

Hence \( m(A_i) = 0 \) and so \( w_i \geq 0 \) a.e. for \( i = 0, 1, \ldots, n \). \( \square \)

**Note.** Let \( T \) be defined as in Theorem 3. If \( T(1) = 1 \) and if \( T \) preserves integrals, then \( T \) need not be nonnegative even for the case \( n = 1 \). Simply take \( w_0 = 3\chi_{[0,1/2]} - \chi_{[1/2,1]} \) and \( w_1 = 3\chi_{[1/2,1]} - \chi_{[0,1/2]} \).

Let \( S_i \) be the closed support of \( \varphi_i \) and let \( V_i \) be a closed subinterval of \( S_i \) such that \( m(V_i) > 0 \) for \( i = 0, 1, \ldots, n \). Define \( Q_n : L^1(I) \to \Phi_n \) by

\[
Q_n(f) = \sum_{i=0}^{n} \left\langle f, \frac{1}{m(V_i)} \chi_{V_i} \right\rangle \varphi_i \quad \text{for each} \quad f \in L^1(I).
\]

Then \( Q_n \) satisfies the conditions in (1) and (2) of Theorem 1. We wish to find \( V_0, V_1, \ldots, V_n \) such that \( Q_n \) is an averaging operator from \( L^1(I) \) to \( \Phi_n \).

**Example 4** Set \( w_i = \frac{1}{m(S_i)} \chi_{S_i} \) for \( i = 0, 1, \ldots, n \). Define \( \alpha_n : L^1(I) \to \Phi_n \) by

\[
\alpha_n(f) = \sum_{i=0}^{n} \langle f, w_i \rangle \varphi_i \quad \text{for each} \quad f \in L^1(I).
\]

Using Theorem 3, it is easy to check that \( \alpha_n \) is an averaging operator from \( L^1(I) \) to \( \Phi_n \). Clearly \( w_i \geq 0 \) and \( \langle 1, w_i \rangle = 1 \) for \( i = 0, 1, \ldots, n \). Also, \( w_0 + 2 \sum_{i=1}^{n-1} w_i + w_n = 2n \) except at the points \( \{ x_1, \ldots, x_{n-1} \} \).
Note. $\alpha_n$ was first constructed in [1] to calculate fixed densities of Frobenius-Perron operators associated with chaotic interval mappings.

Example 5 Let $W_0 = [0, h/2]$, $W_n = [1-h/2, 1]$ and $W_i = [x_i-h/2, x_i+h/2]$ for $i = 1, \ldots, n-1$. Set $w_i = \frac{1}{m(W_i)}\chi_{W_i}$ for $i = 0, 1, \ldots, n$. Define $\beta_n : L^1(I) \to \Phi_n$ by
\[
\beta_n(f) = \sum_{i=0}^{n} \langle f, w_i \rangle \varphi_i \quad \text{for each} \quad f \in L^1(I).
\]
Using Theorem 3, it is easy to check that $\beta_n$ is an averaging operator from $L^1(I)$ to $\Phi_n$. Clearly $w_i \geq 0$ and $\langle 1, w_i \rangle = 1$ for $i = 0, 1, \ldots, n$. Also, $w_0 + 2 \sum_{i=1}^{n-1} w_i + w_n = 2n$ except at the points $\{x_0 + h/2, \ldots, x_{n-1} + h/2\}$.

Note. It has been shown [3] that $\beta_n f$ is a better approximation to $f \in L^1(I)$ than $\alpha_n f$.

Let $V_0, \ldots, V_n$ and $Q_n$ be as above. If $Q_n$ is integral preserving and if $E$ is a subinterval of $[x_k, x_{k+1}]$ and $0 \leq k < n$, then
\[
m(E) = \langle \chi_E, 1 \rangle = \langle Q_n(\chi_E), 1 \rangle = \sum_{i=0}^{n} \left\langle \chi_E, \frac{1}{m(V_i)} \chi_{V_i} \right\rangle \langle \varphi_i, 1 \rangle
\]
and so
\[
m(E) = \frac{m(E \cap V_k) m(S_k)}{m(V_k)} + \frac{m(E \cap V_{k+1}) m(S_{k+1})}{m(V_{k+1})}.
\]
(1)

If $Q_n$ is an averaging operator from $L^1(I)$ to $\Phi_n$, then we will show that either $Q_n = \alpha_n$ or $Q_n = \beta_n$.

Lemma 6 Let $V_0, \ldots, V_n$ and $Q_n$ be as above. Assume $Q_n$ is an averaging operator from $L^1(I)$ to $\Phi_n$. If $m(V_i \cap V_{i+1}) = 0$ for $i = 0, 1, \ldots, n - 1$, then $Q_n = \beta_n$.

Proof. Assume $m(V_i \cap V_{i+1}) = 0$ for $i = 0, 1, \ldots, n - 1$. By Theorem 2, it follows that $m(V_k) = \langle \varphi_k, 1 \rangle$ for $0 \leq k \leq n$. Thus, $m(V_0) = m(V_n) = h/2$ and $m(V_i) = h$ for $i = 1, \ldots, n - 1$. It follows that $V_i = W_i$ for $i = 1, \ldots, n$ where $W_0, W_1, \ldots, W_n$ are as in Example 2. Hence, $Q_n = \beta_n$. □

Lemma 7 Let $V_0, \ldots, V_n$ and $Q_n$ be as above. Assume $Q_n$ is an averaging operator from $L^1(I)$ to $\Phi_n$. If $0 \leq k < n$ and if $m(V_k \cap V_{k+1}) > 0$, then $V_k = S_k$ and $V_{k+1} = S_{k+1}$.
PROOF. Let $0 \leq k < n$ and assume $m(V_k \cap V_{k+1}) > 0$. Applying equation (1) with $E = V_k \cap V_{k+1}$, we see that

$$m(V_k \cap V_{k+1}) = \frac{m(V_k \cap V_{k+1})}{m(V_k)} \cdot m(S_k) \cdot \frac{1}{2} + \frac{m(V_k \cap V_{k+1})}{m(V_{k+1})} \cdot m(S_{k+1}) \cdot \frac{1}{2}.$$ 

It follows that $2 = m(S_k)/m(V_k) + m(S_{k+1})/m(V_{k+1})$ and so $V_k = S_k$ and $V_{k+1} = S_{k+1}$. □

Lemma 8 Let $V_0, \ldots, V_n$ and $Q_n$ be as above and assume $n > 1$. Assume $Q_n$ is an averaging operator from $L^1(I)$ to $\Phi_n$. If $V_k = S_k$ for some $0 < k < n$, then $Q_n = \alpha_n$.

PROOF. Let $0 < j < n$ and assume $V_j = S_j$. Applying equation (1) with $E = [x_j, x_{j+1}]$, we see that

$$m([x_j, x_{j+1}]) = \frac{m([x_j, x_{j+1}])}{2} + \frac{m([x_j, x_{j+1}] \cap V_{j+1})}{m(V_{j+1})} \cdot m(S_{j+1}) \cdot \frac{1}{2}.$$ 

Hence, $m(V_j \cap V_{j+1}) \geq m([x_j, x_{j+1}] \cap V_{j+1}) > 0$. By Lemma 7, $V_{j+1} = S_{j+1}$. By a similar argument, we see that $V_{j-1} = S_{j-1}$. Thus if $V_k = S_k$ for some $0 < k < n$, then $V_i = S_i$ for $i = 0, 1, \ldots, n$ and so $Q_n = \alpha_n$. □

Theorem 9 Assume $Q_n$ is an averaging operator from $L^1(I)$ to $\Phi_n$. Then either $Q_n = \alpha_n$ or $Q_n = \beta_n$.

PROOF. Suppose $Q_n \neq \beta_n$. By Lemma 6, we may choose $k$ such that $m(V_k \cap V_{k+1}) > 0$ and such that $0 \leq k < n$. By Lemma 7, we have $V_k = S_k$ and $V_{k+1} = S_{k+1}$. If $n = 1$, then $V_0 = S_0$ and $V_1 = S_1$ and so $Q_n = \alpha_n$. Suppose $n > 1$. By Lemma 8, we have $Q_n = \alpha_n$ since either $0 < k < n$ and $V_k = S_k$ or $0 < k + 1 < n$ and $V_{k+1} = S_{k+1}$. □

4 Some Averaging Operators on $L^1(I \times I)$

We use the standard Kuhn triangulation of the domain $I \times I$. Divide the square $I \times I$ into $n^2$ equal sub-squares $I_i \times I_j = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ with area $h^2 = 1/n^2$. Then divide each $I_i \times I_j$ into two simplicies

$$\text{co}\{(x_{i-1}, y_{j-1}), (x_i, y_{j-1}), (x_i, y_j)\}, \quad \text{co}\{(x_{i-1}, y_{j-1}), (x_i, y_{j-1}), (x_i, y_j)\},$$

where $\text{co} A$ denotes the convex hull of the set $A$. Thus, we obtain a triangulation $T_h$ of $I \times I$ into a family of $2n^2$ triangles and each triangle has area $h^2/2$. 

6
Let $\Delta_h$ be the space of continuous piecewise linear functions associated with the triangulation $T_h$. Let $\varphi_{ij}$ be the unique function in $\Delta_h$ such that $\varphi_{ij}$ is 1 at the node $(x_i, y_j)$ and 0 at all the other nodes of $T_h$. The $(n + 1)^2$ nodal functions $\{\varphi_{ij}\}_{i,j=0}^{n}$ form a canonical basis for $\Delta_h$ and $\sum_{i=0}^{n} \sum_{j=0}^{n} \varphi_{ij} = 1$.

Let $T$ be a continuous linear operator from $L^1(I \times I)$ to $\Delta_h$. There exist $w_{ij} \in L^\infty(I \times I)$ for $0 \leq i, j \leq n$ such that

$$T(f) = \sum_{i=0}^{n} \sum_{j=0}^{n} \langle f, w_{ij} \rangle \varphi_{ij} \quad \text{for each} \quad f \in L^1(I \times I).$$

Again Theorem 1(2) can be strengthened. As before one can show that if $T$ is nonnegative then $w_{ij} \geq 0$ a.e. for $0 \leq i, j \leq n$. Besides, like Theorem 3 (3), $T$ preserves integrals if and only if

$$\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{2w_{i-1,j-1} + w_{i,j-1} + w_{i-1,j} + 2w_{ij}}{6} = 1 \quad \text{a.e..}$$

Let $S_{ij}$ be the closed support of $\varphi_{ij}$ and let $V_{ij}$ be a closed convex subset of $S_{ij}$ such that $m(V_{ij}) > 0$ for $0 \leq i, j \leq n$. Define $Q_h : L^1(I \times I) \to \Delta_h$ by

$$Q_h(f) = \sum_{i=0}^{n} \sum_{j=0}^{n} \left\langle f, \frac{1}{m(V_{ij})} \chi_{V_{ij}} \right\rangle \varphi_{ij} \quad \text{for each} \quad f \in L^1(I \times I).$$

Then $Q_h$ satisfies the conditions in (1) and (2) of Theorem 1. We wish to find $\{V_{ij}\}_{i,j=0}^{n}$ such that $Q_h$ is an averaging operator from $L^1(I \times I)$ to $\Delta_h$, that is, $Q_h$ satisfies the condition (3) in Theorem 1.

**Example 10** Set $w_{ij} = \frac{1}{m(S_{ij})} \chi_{S_{ij}}$ for $0 \leq i, j \leq n$. Define $\alpha_h : L^1(I \times I) \to \Delta_h$ by

$$\alpha_h(f) = \sum_{i=0}^{n} \sum_{j=0}^{n} \langle f, w_{ij} \rangle \varphi_{ij} \quad \text{for each} \quad f \in L^1(I \times I).$$

Using Theorem 1, it is easy to check that $\alpha_h$ is an averaging operator from $L^1(I \times I)$ to $\Delta_h$.

**Note.** The numerical scheme $\alpha_h$ was developed in [4] to compute absolutely continuous invariant measures associated with two dimensional transformations.

Now the question is whether we can construct an averaging operator $Q_h$ such that $m(V_{ij} \cap V_{kl}) = 0$ whenever $(i, j) \neq (k, l)$. Because of Theorem 2, all boils down to finding $\{V_{ij}\}_{i,j=0}^{n}$ such that $\langle \varphi_{ij}, 1 \rangle = m(V_{ij})$ for each $0 \leq i, j \leq n$. The answer is yes, but we first show that a most intuitive construction of
\( \{V_{ij}\}_{i,j=0}^n \) fails. Let
\[
V_{ij} = (I \times I) \cap \left( \left[ x_i - \frac{h}{2}, x_i + \frac{h}{2} \right] \times \left[ y_j - \frac{h}{2}, y_j + \frac{h}{2} \right] \right), \quad 0 \leq i, j \leq n.
\]

But the corresponding \( Q_h \) fails to be integral preserving. It fails at the four corner nodes. For example, \( m(V_{nn}) = \frac{h^2}{4} \), but \( \langle \varphi_{nn}, 1 \rangle = \frac{h^2}{3} \). Hence by Theorem 2, \( Q_h \) is not an averaging operator.

It turns out that a correct approach is to use a centroid of each triangle in \( T_h \). We construct \( W_{ij} \) as the convex hull of the centroids of the triangles in \( S_{ij} \). The construction of \( W_{ij} \) is shown in Figure 1.

![Fig. 1. Partitioning of the Unit Square for \( n = 4 \)](image_url)

**Example 11** Set \( w_{ij} = \frac{1}{m(W_{ij})} \chi_{W_{ij}} \) for \( 0 \leq i, j \leq n \). Define \( \beta_h : L^1(I \times I) \to \Delta_h \) by
\[
\beta_h(f) = \sum_{i=0}^{n} \sum_{j=0}^{n} \langle f, w_{ij} \rangle \varphi_{ij} \quad \text{for each } f \in L^1(I \times I).
\]

Now we prove that \( \beta_h \) is an averaging operator from \( L^1(I \times I) \) to \( \Delta_h \).

**Note.** From the theoretical analysis in [3] and the fact that each \( W_{ij} \) is a subset of \( S_{ij} \) with much smaller area, one can see that the numerical method based on \( \beta_h \) has a better convergence property than \( \alpha_n \) in the computation of two dimensional absolutely continuous invariant measures; see [2] for more details on approximations of invariant measures.

**PROOF.** By Theorem 2, it suffices to show that \( \langle \varphi_{ij}, 1 \rangle = m(W_{ij}) \) for \( 0 \leq i, j \leq n \). There are four cases to consider.
Case 1 \((i, j) = (0, 0)\) or \((i, j) = (n, n)\).
We consider the case \((i, j) = (n, n)\). Notice that \(\langle \varphi_{nn}, 1 \rangle = h^2/3\). From Figure 2 we see that \(W_{nn}\) is a pentagon \(ABCEF\) and it is made of the square \(ABDF\) of dimension \(h/2\) by \(h/2\) and two congruent triangles \(BCD\) and \(DEF\) whose base and height are \(h/2\) and \((h/2 - h/3)\), respectively. Hence

\[
m(W_{nn}) = \frac{h}{2} \cdot \frac{h}{2} + 2 \cdot \frac{1}{2} \cdot \frac{h}{2} \left( \frac{h}{2} - \frac{h}{3} \right) = \frac{h^2}{3}.
\]

Case 2 \((i, j) = (0, n)\) or \((i, j) = (n, 0)\).
We consider the case \((i, j) = (0, n)\). Notice that \(\langle \varphi_{0n}, 1 \rangle = h^2/6\). As in Case 1 one can show that

\[
m(W_{0n}) = \frac{h}{3} \cdot \frac{h}{3} + 2 \cdot \frac{1}{2} \cdot \left( \frac{h}{2} - \frac{h}{3} \right) = \frac{h^2}{6}.
\]

Case 3 \(1 \leq i, j \leq n - 1\) (Interior Nodes).
Notice in this case that \(\langle \varphi_{ij}, 1 \rangle = h^2\). From Figure 3, we see that \(W_{ij}\) is a hexagon \(ABCDEF\) and it is made of the parallelogram \(BCEF\), whose base is \(h\) and height is \(2h/3\), and two congruent triangles \(ABF\) and \(CDE\) whose base and height are \(h\) and \(h/3\), respectively. Thus

\[
m(W_{ij}) = h \cdot \frac{2}{3}h + 2 \cdot \frac{1}{2} \cdot h \cdot \frac{h}{3} = h^2.
\]

Case 4 All other cases (Boundary Nodes except Four Corner Nodes).
Fig. 3. Interior Case

Notice that in this case $\langle \varphi_{i,j}, 1 \rangle = h^2/2$. As in Case 3 one can verify that
\[
\begin{align*}
m(W_{ij}) &= h \cdot \frac{h}{3} + \frac{1}{2} \cdot h \cdot \frac{1}{3} h = \frac{h^2}{2}.
\end{align*}
\]

So by Theorem 2, $\beta_h$ is an averaging operator. □

**Note.** It is an open question whether if $Q_h$ is an averaging operator from $L^1(I \times I)$ to $\Delta_h$ then $Q_h = \alpha_h$ or $Q_h = \beta_h$.

**References**


